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Green's Function Approach to Quantum Field Theory in Locally Convex Space and Question of Uniqueness

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Abstract

If the set of many point functions $((G_2 - G^0, G_4, \ldots, G_N))$ satisfies the set of equations arising in the ϕ^4 model of quantum field theory, then for a given G_N the set $((G_2 - G^0, G_4, \ldots, G_{N-2}))$, is unique in the domain

$$V = \{((G_2 - G^0, G_4, \ldots, G_{N-2})) : |((G_2 - G^0, G_4, \ldots, G_{N-2}^{\mathrm{irred}}))|_\iota < f_\iota g^{-1/4}\ \forall\ \iota \in \mathfrak{J}\}$$

in a locally convex space equipped with a directed family of semi-norms, where f_ι are positive numbers that depend on details of G_N, and $g \ll 1$ is the effective coupling constant.

1. *Introduction*

In this paper we consider the descending problem in Green's function approach to quantum field theory. By Green's function approach we mean that once equations for Green's functions (many point functions) have been derived from a given Lagrangian one can forget the under-lying operator algebra and one can deal exclusively with Green's functions. By descending problem we mean the problem of finding lower Green's functions for a given N point function $(2 < N < \infty)$. As has been stated in our previous paper (Yoshimura, 1974) one cannot solve a descending problem from $G_N (N \geqslant 6)$ in the ϕ^4 model of quantum field theory by any known procedure. Now let us suppose that a set of Green's functions $((G_n (n \leqslant N < \infty))$ were known or substituted by model functions subject to appropriate conditions and ask whether there can be $G_n' \neq G_n (n \leqslant N - 2)$ corresponding to the given G_N.

Traditionally the quantum mechanics is formulated in Hilbert spaces. But, fortunately or unfortunately, one cannot search Green's functions of quantum field theory within a Hilbert space because Green's functions are not square integrable. Therefore one has to deal with operator equations in more general function spaces.

Recently, Gandac (1973) and Marinescu (1969, 1972) developed powerful theorems for non-linear operator equations in locally convex space. So we consider the descending problems in Green's function approach to quantum field theory in locally convex space instead of Banach space. We then find that $((G_n; 2 \leqslant n \leqslant N-2))$ with semi-norms smaller than certain bounds are unique for a given G_N in the ϕ^4 model.

2. *Notations and Conventions*

Let us take the following Lagrangian density

$$\mathfrak{L} = -\frac{1}{2}\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} - \tfrac{1}{2}m^2\phi^2 - g_0\phi^4 \tag{2.1}$$

and define G_2 as a two-point function in the Heisenberg representation:

$$G_2(x, x') = \langle 0 \,|\, T\phi(x)\phi(x') \,|\, 0 \rangle \tag{2.2}$$

For G_n $(n > 4)$ we take the amputated n-point functions and define G_n^{irred} $(n \geqslant 6)$ as parts of n-point functions that are one-particle-irreducible in any

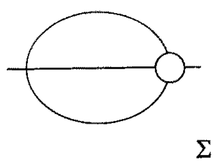

Fig. 1. Graphical representation of Σ.

channel. Then the proper self-energy part Σ is written in terms of G_2 and G_4 as follow (in momentum representation):

$$\Sigma(p^2) = g_0^2 \int dq_1 dq_2 G_2(q_1)G_2(q_2)G_2(p - q_1 - q_2)G_4(q_1, q_2, p - q_1 - q_2, p)$$

$$\equiv [G_2^{*3} * G_4](p^2) \tag{2.3}$$

We renormalise this expression as follows (Taylor & Yoshimura, 1973):

$$\overset{\text{ren}}{\Sigma}(p^2) = \int_{m_r^2}^{p^2} d(p'^2) \int_{m_r^2}^{p'^2} d(p''^2) \frac{d^2}{d(p''^2)} [G_2^{*3} * G_4](p''^2) \equiv [\![G_2^{*3} * G_4]\!](p^2) \tag{2.4}$$

Here G_4 is so normalised that it is equal to 1 for zero momenta, so that our g corresponds to g^2 in the more conventional normalisation. The $*$ is short-hand for convolution.

Four-point function has the following structure:

$$G_4 = 1 + g_0[G_2^{*2} * G_4] + g_0[G_2^{*4} * G_4^{*2}] + g_0[G_2^{*3} * G_6^{\text{irred}}] \qquad (2.5)$$

which we renormalise by substitution

$$g_0[G_2^{*2} * G_4](p_1, p_2, p_3, p_4) \to g^{1/2} \prod_{i=1}^{4} D^{-1}(p_i) D(p_i) [G_2^{*2} * G_4](p_1, \ldots, p_4)$$

$$\equiv [\![G_2^{*2} * G_4]\!](p_i, \ldots, p_4) \qquad (2.6)$$

etc., where

$$D(p_i) = p_{i\mu} \frac{\partial}{\partial p_{i\mu}} \qquad (2.7)$$

and the operation D^{-1} is carried out in such a way that the resulting expression is equal to zero at zero momenta (0, 0, 0, 0) (Taylor & Yoshimura, 1973).

As Σ is not bounded for $|p^2| \to \infty$ and G_2 has a pole at $p^2 = m_r{}^2$, it is more convenient to deal with σ defined as follows:

$$\sigma^{(p^2)} = \frac{\Sigma^{\text{ren}}(p^2)}{(p^2 - m_r{}^2 - g\,\Sigma^{\text{ren}}\,(p^2))(p^2 - m_r{}^2 - i\epsilon)} \qquad (2.8)$$

Then the equations to be considered are:

$$\sigma_2 - \Gamma_2(G_4, \sigma) = 0 \qquad (2.9)$$

$$G_4 - 1 - \Gamma_4(G_6^{\text{irred}}, G_4, \sigma) = 0 \qquad (2.10)$$

where

$$\Gamma_2(G_4, \sigma) = \frac{gG^0 [\![(G^0 + \sigma)^{*3} * G_4]\!]}{\{(G^0)^{-1} - g[\![(G^0 + \sigma)^{*3} * G_4]\!]\}} \qquad (2.11)$$

$$\Gamma_4(G_6^{\text{irred}}, G_4, \sigma) = g^{1/2} [\![(G^0 + \sigma)^{*2} * G_4]\!] + g[\![(G^0 + \sigma)^{*4} * G_4{}^{*2}]\!] + g[\![(G^0 + \sigma) +^{*3} * G_6^{\text{irred}}]\!] \qquad (2.12)$$

Then the question is whether these equations can be satisfied by more than one set of $((\sigma, G_4))$ for a given G_6^{irred}. Let us define a pair of operators

$$((T[G_4, \sigma\colon \cdot, \cdot], U[G_6^{\text{irred}}, G_4, \sigma\colon \cdot, \cdot])) \equiv U \qquad (2.13)$$

where

$$T[G_4, \sigma\colon s, t] =$$

$$= \frac{gG^0([\![(G^0 + \sigma)^{*3} * G_4]\!] - [\![(G^0 + \sigma + s)^{*3} * (G_4 + t)]\!])}{\{(G^0)^{-1} - g[\![(G^0 + \sigma)^{*3} * G_4]\!]\}\{(G^0)^{-1} - g[\![(G^0 + \sigma + s)^{*3} * (G_4 + t)]\!]\}} - s \qquad (2.14)$$

$$U[G_6^{\text{irred}}, G_4, \sigma, s, t]$$
$$= g^{1/2}[\![(G^0 + \sigma + s)^{*2} * (G_4 + t)]\!] + g[\![(G^0 + \sigma + s)^{*4} * (G_4 + t)^{*2}]\!]$$
$$+ g[\![(G^0 + \sigma + s)^{*3} * G_6^{\text{irred}}]\!] - g^{1/2}[\![(G^0 + \sigma)^{*2} * G_4]\!]$$
$$-g[\![(G^0 + \sigma)^{*4} * G_4^{*2}]\!] - g[\![(G^0 + \sigma)^{*3} * G_6^{\text{irred}}]\!] - t \tag{2.15}$$

Then the question becomes whether the equation

$$U[((s, t))] = 0 \tag{2.16}$$

has a non-trivial solution $((s, t)) \neq 0$. From now on we call this problem Q_6. (And similarly, we can define problems $Q_n, n = 4, 8, 10. \ldots$)

3. *Locally Convex Space*

As a space more general than Hilbert space, we consider a locally convex space.

Semi-norm is a map $|\cdot|$ from a vector space $\mathfrak{B}$ over a field $\mathfrak{K}$ into $\mathbb{R}^+ \cap \{0\}$ satisfying the following conditions:

(SN1) $\quad |x_1 + x_2| \leqslant |x_1| + |x_2| \qquad \forall x_1, x_2 \in \mathfrak{B}$

(SN2) $\quad |ax| = |a||x| \qquad \forall a \in \mathfrak{K}, \qquad \forall x \in \mathfrak{B}$

If convergence in $\mathfrak{B}$ is defined in terms of a directed family of semi-norms satisfying conditions

(SN4) $\quad \forall x \in \mathfrak{B}, \quad x \neq 0 \quad \exists \iota \in \mathfrak{J} \quad$ such that $\quad |x|_\iota \neq 0$

(SN4) $\quad \forall \iota_1 \in \mathfrak{J} \quad \forall \iota_2 \in \mathfrak{J} \quad \exists \iota \in \mathfrak{J} \quad$ such that

$$|x|_{\iota_1} \leqslant |x|_\iota, |x|_{\iota_2} < |x|_\iota$$

then we call $\mathfrak{B}$ a locally convex space. A transfinite set $\mathfrak{J} = \{\iota, <\}$ is said to be directed if

$$\forall \iota_1, \iota_2 \in \mathfrak{J} \qquad \exists \iota \in \mathfrak{J} \qquad \text{such that} \qquad \iota > \iota_1, \iota > \iota_2$$

We now have the following theorem adapted from Gandac's paper and Marinescu's book.

[*Theorem* 1. Let f be a map from a set U in a locally convex complete space $\mathfrak{C}$ with a topology given by a directed family of semi-norms, where

$$U = \{x : x \in \mathfrak{C}, \qquad |x - x_0|_\iota \leqslant R_\iota \qquad \forall \iota \in \mathfrak{J}\} \tag{3.1}$$

And let the Fréchet derivative $f'(x)$ $(x \in U)$ be a (linear) map from U into $\mathfrak{L}(\varphi, \mathfrak{C})$, and let the following conditions be satisfied:

1) $$|f'(x) - f'(y)|_{\iota,\varphi(\iota)} = L_\iota |x - y|_{\varphi(\iota)}$$

2) f' is inversible at $x = x_0$:

$$|f'(x_0)]^{-1} \in \mathfrak{L}(\varphi^{-1}, \mathfrak{C}) \tag{3.2}$$

3) $$|[f'(x_0)]^{-1} f(x_0)|_\iota < \eta_\iota \qquad \forall\, \iota \in \mathfrak{J} \tag{3.3}$$

4) $$|[f'(x)]^{-1}|_{\iota,\varphi^{-1}(\iota)} < b_\iota \qquad \forall\, \iota \in \mathfrak{J} \tag{3.4}$$

5) $$0 < h_\iota \equiv \eta_\iota b_\iota L_{\varphi^{-1}(\iota)} < \tfrac{1}{2} \tag{3.5}$$

6) $$R_\iota \geqslant r_\iota = (1 - \sqrt{(1 - 2h_\iota)})\eta_\iota h_\iota^{-1} \tag{3.6}$$

Then the solution of the equation $f(x) = \theta$ is unique and the Newton-Kantorovich type successive approximation

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n) \qquad n \in \mathbb{R}^+ \cup \{0\} \tag{3.7}$$

converges to the unique solution of the equation $f(x) = \theta$ in U. Here the semi-norms of linear operators are defined as follows

$$|T|_{\iota,k} = \sup_{|x|_k > 0} |Tx|_\iota / |x|_k \qquad \iota, x \in \mathfrak{J} \tag{3.8}$$

Now, let us apply this theorem to our problem Q_6. The Fréchet derivative of the map U can be written in the form of a super matrix

$$U'[G_6^{\text{irred}}, G_4, \sigma; s, t] = \begin{bmatrix} T_{,1}[G_4, \sigma; s, t] & T_{,2}[G_4, \sigma; s, t] \\ U_{,1}[G_6^{\text{irred}}, G_4, \sigma; s, t] & U_{,2}[G_6^{\text{irred}}, G_4, \sigma; s, t) \end{bmatrix} \tag{3.9}$$

where

$$T_{,1}[G_4, \sigma; s, t] = \frac{3g[\![(G^0 + \sigma + s))^{*2} * \cdot * (G_4 + t)]\!]}{\{(G^0)^{-1} - g[\![(G^0 + \sigma + s)^{*3} * (G_4 + t)]\!]\}^2} - I \tag{3.10}$$

$$T_{,2}[G_4, \sigma; s, t] = \frac{g[\![(G^0 + \sigma + s)^{*3} * \cdot]\!]}{\{(G^0)^{-1} - g[\![(G^0 + \sigma + s)^{*3} * (G_4 + t)]\!]\}^2} \tag{3.11}$$

$$\begin{aligned} U_{,1}[G_6^{\text{irred}}, G_4, \sigma, s, t] &= 2g^{1/2}[\![(G^0 + \sigma + s) * \cdot * (G_4 + t)]\!] \\ &\quad + 4g[\![(G^0 + \sigma + s)^{*3} * \cdot * (G_4 + t)^{*2}]\!] \\ &\quad + 3g[\![(G^0 + \sigma + s)^{*2} * \cdot * G_6^{\text{irred}}]\!] \end{aligned} \tag{3.12}$$

$$\begin{aligned} U_{,2}[G_6^{\text{irred}}, G_4, \sigma; s, t] &= g^{1/2}[\![(G^0 + \sigma + s)^{*2} * \cdot]\!] \\ &\quad + 2g[\![(G^0 + \sigma + s)^{*4} * (G_4 + t) * \cdot]\!] - I \end{aligned} \tag{3.13}$$

Here, numerical factors before g indicate the number of terms containing the respective numbers of factors $G^0 + \sigma + s$ etc.; for orders of convolution one has to refer to Fig. 2.

$$[\forall\, \iota \in \mathfrak{J}' \subseteq \mathfrak{J}, |((\sigma, G_4))|_\iota < \infty] \Rightarrow [\sup_{p^2 \in \mathbb{R}} |\sigma(p^2)| < \infty] \tag{3.14}$$

$$[\forall\, \iota \in \mathfrak{J}'' \subseteq \mathfrak{J}, |((\sigma, G_4))|_\iota < \infty] \Rightarrow$$

$$[|G_4(p_1, p_2, p_3, p_4)| \left(\sum_{i=1}^{4} \log(|p_i^2|/m_r^2) \right) \log \left(\sum_{i=1}^{4} \log(|p_i^2|/m_r^2) \right)$$

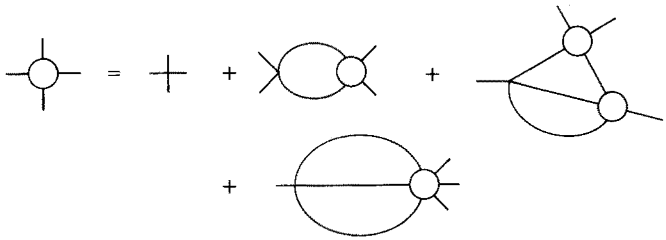

Fig. 2. Graphical representation of the equation (2.5).

$$\ldots\left(\log\left(\log\left(\ldots\left(\sum_{i=1}^{4}\log\left(|p_i^2|/m_r^2\right)\right)\ldots\right)\right)\right)^2<\infty$$
$$\left.\text{for } |p_i^2|\to\infty\right] \qquad (3.15)$$

[The second condition (3.15) may be replaced by another condition of similar nature. Let us call this locally convex complete space of pairs $((\sigma, G_4))$ (not subject to the equations (2.9) and (2.10)) equipped with this family of semi-norms $\Omega[\mathfrak{J}]$. We do not know, however, whether one can choose $\mathfrak{J}'$ and $\mathfrak{J}''$ so that $\mathfrak{J}' \cap \mathfrak{J}'' = \phi$. It should be noticed that the map $\Gamma_2 \otimes \Gamma_4$ form $\Omega[\mathfrak{J}]$ into itself is neither completely continuous nor non-expansive, so that unfortunately Reich's theory (Reich, 1972) is not applicable.

Let us define a domain V in $\Omega[\mathfrak{J}]$:

$$V=\{((s,t)): ((s,t))\in\Omega[\mathfrak{J}], \quad |((s,t))|_\iota<R_\iota \quad \forall\,\iota\in\mathfrak{J}\} \qquad (3.16)$$

where R_ι are positive numbers $\lesssim O(g^{-1/4})$. Then

$$|U'[((s_1,t_1))]-U'[((s_2,t_2))]|_{\iota,\varphi(\iota)}<L_\iota|((s_1,t_1))-((s_2,t_2))|_\iota \qquad (3.17)$$

with $L_\iota=O(g^0)$, where φ is a bijection from $\mathfrak{J}$ to itself.

Then one finds that $U'[((s_0,t_0))]$ is inversible on any $((s_0,t_0))$ such that for any $\iota\in\mathfrak{J}$, $|((s_1,t_1))|_\iota<O(g^{1/2})$, and

$$|[U'[((s_0,t_0))]]^{-1}U[((s_0,t_0))]|_\iota<\eta_\iota\leqslant O(g^{1/2}) \qquad (3.18)$$

$$|[U'[((s_0,t_0))]]^{-1}|_{\iota,\varphi^{-1}(\iota)}<b_\iota=O(g^0) \qquad (3.19)$$

and

$$0<h_\iota=\eta_\iota b_\iota L_{\varphi^{-1}(\iota)}=O(g^{1/2})<\tfrac{1}{2} \qquad (3.20)$$

$$R_\iota\geqslant r_\iota=(1-\sqrt{(1-2h_\iota)})\eta_\iota h_\iota^{-1}=O(g^{1/2}) \qquad (3.21)$$

Therefore the solution of the problem Q_6 is unique in V. If a non-trivial solution of equation (2.16) exists at all, it must have at least one large semi-norm. But the snag is that the domain of such $((s,t))$ is not convex so that one cannot apply any known method to search for a 'large' solution.

Similarly, for $Q_N, N=8, 10, \ldots<\infty$, one finds that 'small' solution is unique for a given G_N.

4. *Problem* Q_4

Let us call Q_4 the problem with equation (2.9) with a given G_4. This problem has a structure different from those of $Q_N (N \geqslant 6)$.

For this problem, Theorem 1 is applicable and one finds that the solution is unique up to $O(g^{-1/3})$.

The first approximation

$$\sigma_1 = [I - \Gamma_2'[0]]^{-1}\Gamma_2[0] \tag{4.1}$$

gives the graphs of Fig. 4. Therefore all the thresholds appear in σ_1 though coefficients are modified by higher approximations. As to the lower bound

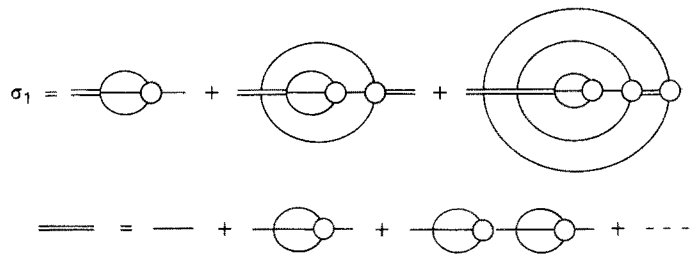

Fig. 3.–Graphical representation of the first approximation σ_1 to Q_4. Here ——— stands for G^0.

of semi-norms, we have the following theorem (Gandac, 1973).

[*Theorem* 2. Let the conditions

$$0 < r_\iota \leqslant R_\iota \tag{4.2}$$

$$0 \leqslant \sup_{x \in S(x_0, \{r_\iota\})} |[F'(x_0)]^{-1}[F'(x) - F'(x_0)]|_{\iota,\iota} < q_\iota < 1 \tag{4.3}$$

$$a_\iota = |[F'(x_0)]^{-1} F x_0|_\iota \leqslant r_\iota (1 - q_\iota) \tag{4.4}$$

be satisfied for any $\iota \in \mathfrak{J}$.

Then the equation $Fx = \theta$ has a unique solution $x^\blacktriangle$ in $S(x_0, \{r_\iota\})$ and $|x^\blacktriangle - x_0|_\iota$ is bounded as follows

$$\frac{a_\iota}{1 + q_\iota} \leqslant |x^\blacktriangle - x_0| \leqslant \frac{\iota}{1 - q_\iota} \tag{4.5)]}$$

Now let us apply this theorem to the problem Q_4. Take $\sigma_0 = 0$. Then one finds $q_\iota = O(g)$, $a_\iota = O(g)$ so that $|x^\blacktriangle|_\iota$ must be $O(g)$, too, i.e. semi-norms of $x^\blacktriangle$ cannot be too small or too large. This theorem can be applied also to estimations of higher Newton-Kantorovich approximations. On the other hand, this theorem is trivial for the problems $Q_n (n \geqslant 6)$ because: $a_\iota \equiv 0$.

5. *Concluding Remarks*

Uniqueness of 'small' solution does not, however, imply applicability of perturbation theory, because input is not given in forms of power series in the coupling constant. A question related to this is, What G_4 can be chosen as a input to the infinite system of equations for many point functions whose form is specified by the given Lagrangian?

On the other hand, locally convex space is general enough so that functions that cannot be contained in any locally convex space need not be considered as serious candidates for physically meaningful solutions.

Uniqueness of the descending problem does not, however, imply uniqueness of solution of an ascending problem. By ascending problem we mean the problem of finding higher many-point functions from a given set of lower many-point functions.

Another important question is how to extend the techniques discussed above to the case when the effective coupling constant is not very small. For the imaginary part of σ to remain positive definite, $G_4(p_1, p_2, p_3, p_4)$ has to decrease rapidly as $|p_i^2|$ tend to infinity. Therefore, it seems appropriate to choose a family of semi-norms in such a way that convergence should guarantee the positive definiteness of $\mathrm{Im}\,\sigma$. We hope to come back to this problem in the near future. (As G_4 is complex, σ need not have a real ghost pole even if G_4 does not decrease rapidly.)

If a four-point function with external lines is given instead of an amputated one, the problem Q_4 becomes trivial, but $Q_n\,(n \geqslant 6)$ do not.

For an interaction that gives rise to a two-particle threshold in the self-energy part, the renormalisation procedure (2.4) requires use of pseudo-functions so that positive definiteness of the imaginary part of a two-point function is jeopardised.

References

Gandac, F. (1973). *Studii şi cercetari matematice (Bucureşti)*, **25**, 1473.

Marinescu, G. (1969, 1972). Tratat de analiza functionala, I, II, Bucureşti.

Reich, S. (1972). *Mathematische Zeitschrift*, **125**, 17.

Taylor, J. G. and Yoshimura, T. (1973). *Which Green's Functions Equations*? King's College (London) Preprint.

Yoshimura, T. (1974). Descending problems in Green's function approach to quantum field theory, *Communications in Mathematical Physics*, to be published.